Average value of a random dice roll?

$$\frac{1+2+3+4+5+6}{6} = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + \cdots + 6\cdot\frac{1}{6} = \sum_x x\cdot P(X=x), \quad X = \text{result of a single die roll}$$

If X is an random variable (RV) with probability distribution f , then

- **Mean or expected value of X:**

$$\mu_X = E(X) = \begin{cases} \sum_x x\cdot f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x\cdot f(x) & \text{if } X \text{ is continuous} \end{cases}$$

- **Variance of X:**

$$\sigma_X^2 = E[(X-\mu)^2] = \begin{cases} \sum_x (x-\mu)^2\cdot f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2\cdot f(x) & \text{if } X \text{ is continuous} \end{cases}$$

Practical formula: $\sigma_X^2 = E(X^2) - \mu^2$

- **Standard deviation** of X is the square root σ_X of the variance.

Example 1. A biased coin has probability p of heads. Let X = 1 if it lands heads and X=0 if it lands tails. Find E(X) and σ_X

Answer.

X	0	1
P(X)	1-p	p

$$\mu = E(X) = (0)\cdot(1-p) + (1)\cdot p = p$$

$$E(X^2) = (0)^2\cdot(1-p) + (1)^2\cdot p = p$$

$$\Rightarrow \quad \sigma^2 = p - p^2 = p(1-p)$$

Example 2. A random variable X measures the life in years of a device. Its PDF is $f(x) = 20000/x^3$ if $x > 100$ and $f(x) = 0$ otherwise. Find the expected life of the device. Also find the standard deviation of X.

Answer. $$E(X) = \int_{100}^{\infty} x\cdot\frac{20000}{x^4}\,dx = \int_{100}^{\infty}\frac{20000}{x^3}\,dx = \left[-\frac{10000}{x^2}\right]_{100}^{\infty} = 1 \text{ year}$$

$$E(X^2) = \int_{100}^{\infty} x^2\cdot\frac{20000}{x^4}\,dx = \int_{100}^{\infty}\frac{20000}{x^2}\,dx = \left[-\frac{20000}{x}\right]_{100}^{\infty} = 200$$

$$\Rightarrow \sigma^2 = 200 - 1^2 = 199 \Rightarrow \sigma = \sqrt{199} \approx 14 \text{ years}$$

Combining RVs: $E(X+Y)=E(X)+E(Y)$, $E(c\cdot X)=c\cdot E(X)$, $\sigma_{c\cdot X}=c\cdot\sigma_X$

If X and Y are independent RVs (like X= 1st flip of a coin, Y = 2nd flip of a coin), then variance is also additive:

$$\sigma^2_{X+Y}=\sigma^2_X+\sigma^2_Y$$

Example 3. Let X = # heads when flipping a fair coin 3 times. Find mean and variance of X.

Answer. Let X_i = # heads in i-th flip, so $E(X_i)=(1/2)(1/2)=0.25$ by Example 1.
Then $X=X_1+X_2+X_3$
So $E(X)=E(X_1)+E(X_2)+E(X_3)=1.5$
$\sigma^2_X=\sigma^2_{X_1}+\sigma^2_{X_2}+\sigma^2_{X_3}=0.75$

Special RV's. Suppose a coin is biased with probability p of heads and q of tails.

Binomial RV: If X counts the number of heads in n flips of our coin, then:

$$E(X)=np,\qquad \sigma^2_X=np(1-p),\qquad \sigma_X=\sqrt{np(1-p)}$$

Geometric RV: If Y counts the number of tails when repeatedly flipping our coin before getting a head, then:

$$E(X)=\frac{1-p}{p}=\frac{1}{p}-1,\qquad \sigma_X=\frac{1-p}{p^2}$$

Example 4. You guess all 10 questions on an ABCD multiple choice exam.
(a) Starting with Problem 1, how many wrong guesses do you expect to make before you finally get a correct guess?

Expectation of geometric RV with p=1/4 of success is p=1/(1/4)-1=4-1=3.

(b) How many correct answers do you expect to get?

Expectation of binomial RV with p=1/4 of success is np=10x(1/4)=2.5.

Additional example of discrete RV:

Example 5. American roulette has 18 red spaces, 18 black spaces, and two green spaces. If you bet \$1 on red, what is the expected monetary gain? What is the standard deviation in this gain?

Answer.

X	-\$1	+\$1
P(X)	20/38	18/38

E(X)=-1*(20/38)+1*(18/38)=-\$0.05:

In the long run, you expect to lose 5 cents per play when we bet \$1 each time.

$$E(X^2)=(-1)^2\cdot\frac{20}{38}+(+1)^2\cdot\frac{18}{38}=1\quad\Rightarrow\quad\sigma^2=1-0.05^2=0.9975$$

In the long run, the volatility of each play's monetary return swings by about sqrt(\$0.9975)=\$1 relative to the expected return of -5 cents.